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MODELS OF POLARIZABLE CONTINUA WITH INTERNAL MECHANICAL MOMENTS*

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General methods of constructing models of continua are used to obtain a closed system of equations for a polarizable continuum with internal mechanical moments, and the distribution of small perturbations in such a medium studied.

1. Consider a system of N material points with masses m_ν and radius vectors $\mathbf{r}_\nu$, whose motion relative to the inertial frame of reference is described by the following Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{r}_\nu^{\cdot}} - \frac{\partial L}{\partial \mathbf{r}_\nu} = \mathbf{f}_\nu \tag{1.1}$$

Here L is the Lagrangian of the system of material points, $\mathbf{f}_\nu$ is the external force vector acting on the ν-th point, and t is the time.

Let the Lagrangian be invariant with respect to translation and rotation of the frame of reference, and to the translation of the initial instant of time. We shall also assume in accordance with the Galileo's principle of relativity, that the dependence of the Lagrangian on the velocities $\mathbf{r}_\nu^{\cdot}$ has the following form (here and henceforth the summation over ν is carried out from $\nu = 1$ to $\nu = N$):

$$L = \frac{1}{2}\sum m_\nu \mathbf{r}_\nu^{\cdot 2} - U(t, \mathbf{r}_1, \ldots, \mathbf{r}_N) \tag{1.2}$$

(In this case the passage to another inertial frame of reference changes the Lagrangian by a total derivative of some function). Then the laws of conservation of momentum $\mathbf{\Pi}$, angular momentum $\mathbf{K}$, energy ε and mass momentum of the system $\mathbf{G}$ all hold in the closed system (i.e. when $\mathbf{f}_\nu = 0$).

Taking into account the external forces, we obtain from (1.1) the following balance equations:

$$\begin{gathered}\frac{d\mathbf{\Pi}}{dt} = \sum \mathbf{f}_\nu, \quad \frac{d\mathbf{E}}{dt} = \sum \mathbf{f}_\nu \cdot \mathbf{r}_\nu^{\cdot} \\ \frac{d\mathbf{K}}{dt} = \sum \mathbf{r}_\nu \times \mathbf{f}_\nu, \quad \frac{d\mathbf{G}}{dt} = -t\sum \mathbf{f}_\nu \\ \mathbf{\Pi} \equiv \sum m_\nu \mathbf{r}_\nu^{\cdot}, \quad \mathbf{K} \equiv \sum \mathbf{r}_\nu \times m_\nu \mathbf{r}_\nu^{\cdot} \\ \mathrm{E} \equiv \sum \frac{1}{2} m_\nu \mathbf{r}_\nu^{\cdot 2} + U, \quad G \equiv \sum (m_\nu \mathbf{r}_\nu - t m_\nu \mathbf{r}_\nu^{\cdot})\end{gathered} \tag{1.3}$$

The existence of ten balance equations (1.3) is justified by the corresponding symmetry of the Lagrangian L, noted above, with respect to the complete ten-parametric group of the

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Galileo transformations.

Let $\mathbf{R}$ be the radius vector of some point associated with the system. We can always write the expressions for $\mathbf{K}, \mathbf{G}$ in the form

$$\mathbf{K} = \mathbf{R} \times \mathbf{\Pi} + M\mathbf{k}, \quad \mathbf{G} = M\mathbf{R} - t\mathbf{\Pi} + M\mathbf{l}, \quad M = \sum m_\nu \tag{1.4}$$

$$\mathbf{k} = \frac{1}{M}\sum (\mathbf{r}_\nu - \mathbf{R}) \times m_\nu \dot{\mathbf{r}}_\nu, \quad \mathbf{l} = \frac{1}{M}\sum m_\nu (\mathbf{r}_\nu - \mathbf{R}) \tag{1.5}$$

The quantities $\mathbf{k}$ and $\mathbf{l}$ can be regarded as the internal mechanical characteristics of the system. If $\mathbf{R}$ is the radius vector of the centre of inertia of the system, then from the last expression of (1.5) it follows that $\mathbf{l} \equiv 0$. The non-zero characteristic $\mathbf{l}$ should, generally speaking, be included in the number of defining parameters, provided that the velocity of the mechanical system is defined as the velocity of some point of the system not coinciding with its centre of inertia. (For example, if the velocities of the atoms possessing a dipole moment are defined as the velocities of their nuclei).

Under the Galilean transformation $\mathbf{r} = \mathbf{r}' + \mathbf{V}t'$, $t = t'$, $\mathbf{V} = \text{const}$ the quantities $\mathbf{K}, \mathbf{G}, \mathbf{k}, \mathbf{l}$ will be transformed as follows:

$$\mathbf{K} = \mathbf{K}' + \mathbf{G}' \times \mathbf{V}, \; \mathbf{k} = \mathbf{k}' + \mathbf{l}' \times \mathbf{V}, \; \mathbf{G} = \mathbf{G}', \; \mathbf{l} = \mathbf{l}' \tag{1.6}$$

Moreover, the balance equations (1.3) obtained for a system of material points will be generalized to the case of a continual model. The representations (1.4) will be taken into account and the transformation formulas (1.6) will be regarded as valid for the internal mechanical moments of a continuum.

2. Let us consider a model of a continuum with internal mechanical moments: the internal mass moment $\mathbf{l}$ and internal angular momentum $\mathbf{k}$ relate to unit mass of the medium. We construct the model by postulating an integral law of conservation of mass of the medium, and integral laws representing a generalization of the balance equations (1.3) for a system of material points. The first three equations, i.e. the equation of momentum, energy and angular momentum have the usual form /1/ except for the fact that the moment $\mathbf{p}$ of the unit mass of the medium possessing the internal mass moment $\mathbf{l}$, is not equal to the velocity of the medium $\mathbf{v}$. The equation generalizing the last equation of (1.3) for a continuum is new, and has the form

$$\frac{d}{dt}\int_V d\mathbf{G} = -\int_\Sigma t p^{\alpha\beta} n_\beta \mathbf{э}_\alpha \, d\Sigma + \int_\Sigma Q^{\alpha\beta} n_\beta \mathbf{э}_\alpha \, d\Sigma - \int_V t\mathbf{F}\, dV + \int_V N\, dV$$

Here $d\mathbf{G}$ is the mass moment of an element of the medium of volume dV, by virtue of the second relation of (1.4) $d\mathbf{G} = (\mathbf{r} - t\mathbf{p} + \mathbf{l})\,\rho dV$; ρ is the mass density of the medium, $p^{\alpha\beta}$ are the stress tensor components, F is the density of external forces, $N, Q^{\alpha\beta}$ are the quantities defining the additional mass moment fluxes within the volume V and across its surface Σ; the Greek tensor indices take the values 1, 2, 3 and the Latin indices the values 1, 2, 3, 4.

In the region where the characteristics of the medium change smoothly, the integral laws mentioned above reduce to the differential equations (2.1)–(2.4), and (2.1) must be used to derive (2.3), (2.4) for the internal moments $\mathbf{k}$ and $\mathbf{l}$ from the corresponding integral laws

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho\mathbf{v} = 0, \quad \rho\frac{dp^\alpha}{dt} = \nabla_\beta p^{\alpha\beta} + F^\alpha \tag{2.1}$$

$$\rho\frac{de}{dt} = \nabla_\beta (p^{\alpha\beta} v_\alpha) - \nabla_\beta q^\beta + F_\alpha v^\alpha + F_4 \tag{2.2}$$

$$\rho\frac{dk^\alpha}{dt} = \varepsilon^{\alpha\beta\gamma} p_{\alpha\beta} + \nabla_\beta q^{\alpha\beta} + \rho[\mathbf{p} \times \mathbf{v}]^\alpha + L^\alpha \tag{2.3}$$

$$\rho\frac{dl^\alpha}{dt} = \nabla_\beta Q^{\alpha\beta} + \rho(p^\alpha - v^\alpha) + N^\alpha \tag{2.4}$$

Here e is the total energy of unit mass of the medium, and $F_4, L^\alpha, q^\alpha, q^{\alpha\beta}$ are quantities defining the additional energy and angular momentum fluxes of the medium respectively, within its volume V and across its surface Σ. When $l^\alpha \equiv Q^{\alpha\beta} \equiv N^\alpha \equiv 0$, Eq.(2.4) reduces to the relation $\mathbf{p} = \mathbf{v}$, which is normally used in the mechanics of continua.

In particular, using the quantities $F^\alpha, F_4, L^\alpha, N^\alpha$ we can describe the action of an electromagnetic field obeying Maxwell's equations

$$\operatorname{div}\mathbf{D} = 4\pi q, \quad \operatorname{rot}\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \tag{2.5}$$
$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

on the continuum. Here $q, \mathbf{j}$ are the electric charge and current densities, $\mathbf{E}$ and $\mathbf{D}$ are the electric field strength and induction, $\mathbf{H}$ and $\mathbf{B}$ denote the magnetic field and induction, and c is the velocity of light in vacuo.

To specify the quantities $F^\alpha, F_4, L^\alpha, N^\alpha$, we will use the following argument.

When there are no external forces, (2.1)–(2.4), written for a system which includes, in addition to a continuum, an electromagnetic field, have the following corresponding tensor equations in the relativistic mechanics:

$$\nabla_k P^{ik}=0, \quad \rho \frac{dK^{ij}}{d\tau}=P^{ij}-P^{ji}+\nabla_k q^{ijk} \tag{2.6}$$

Here P^{ik} are components of the four-dimensional energy-momentum tensor of the system, $d\tau$ is the differential of the characteristic time of the particle, K^{ij} are components of the four-dimensional internal mechanical momentum tensor, and we have, in the characteristic frame of reference $K^{*\alpha\beta}=\varepsilon^{\alpha\beta\gamma}k_\gamma^*, K^{*\alpha 4}=cl^{*\alpha}$. The expression for the components P^{ij} can always be written in the form /2/

$$P^{ij}=T^{ij}+S^{ij}_{(M)}, \quad S^{ij}_{(M)}=\frac{1}{16\pi}F_{pq}H^{pq}g^{ij}-\frac{1}{4\pi}F^{i\cdot}_{\cdot p}H^{jp} \tag{2.7}$$

where T^{ij} are components of the four-dimensional energy-momentum tensor of the system, $S_{(M)}{}^{ij}$ are the components of the four-dimensional Minkowskii tensor, and F_{ij} and H_{ij} are the components of four-dimensional electromagnetic field tensors. Taking into account the first relation of (2.7), we can write (2.6) in the form

$$\nabla_k T^{ik}=-\nabla_k S^{ik}_{(M)}; \quad \rho\frac{dK^{ij}}{d\tau}=T^{ij}-T^{ji}+S^{ij}_{(M)}-S^{ji}_{(M)}+\nabla_k q^{ijk}$$

Further, regarding the electromagnetic field as a source of external forces and taking into account relation (2.7) and Maxwell's equations (2.5), we obtain

$$\begin{aligned} &F^\alpha=-\nabla_k S^{\alpha k}_{(M)}=qE^\alpha+\frac{1}{c}\mathbf{j}\times\mathbf{B}^\alpha+\frac{1}{2}(\mathbf{P}\nabla^\alpha\mathbf{E}-\mathbf{E}\nabla^\alpha\mathbf{P}+\\ &\mathbf{M}\nabla^\alpha\mathbf{B}-\mathbf{B}\nabla^\alpha\mathbf{M})\\ &F_4=-c\nabla_k S^{4k}_{(M)}=\mathbf{j}\cdot\mathbf{E}+\mathbf{E}\frac{\partial\mathbf{P}}{\partial t}+\mathbf{B}\frac{\partial\mathbf{M}}{\partial t}-\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{P}+\mathbf{B}\cdot\mathbf{M})\\ &L_\alpha=\varepsilon_{\alpha\beta\gamma}S^{\beta\gamma}_{(M)}=\mathbf{M}\times\mathbf{B}_\alpha+\mathbf{P}\times\mathbf{E}_\alpha\\ &N^\alpha=\frac{1}{c}(S^{\alpha 4}_{(M)}-S^{4\alpha}_{(M)})=\frac{1}{c}\mathbf{P}\times\mathbf{B}^\alpha+\frac{1}{c}\mathbf{E}\times\mathbf{M}^\alpha \end{aligned} \tag{2.8}$$

Equations (2.8) are written in a global Cartesian basis of Minkowskii space with the metric $\|g_{ij}\|=\mathrm{diag}(-1,\ -1,\ -1,\ 1)$, while ∇_k and ∇^α denote, respectively, the four-dimensional and three-dimensional covariant derivatives.

3. Henceforth, we shall confine ourselves to the case of a non-conducting, charge-free medium. We shall also assume that there is no magnetization in the characteristic frame of reference. Then, from the general formulas for transforming the electromagnetic parameters an changing from one inertial frame of reference to another /1/, it follows that

$$\mathbf{M}=c^{-1}\cdot\mathbf{P}\times\mathbf{v},\ \mathbf{P}=\mathbf{P}^* \tag{3.1}$$

apart from terms of order v^2/c^2. An asterisk denotes quantities in the characteristic frame of reference. Using relations (1.6), we obtain the following relations for the internal moments:

$$\mathbf{k}=\mathbf{k}^*+\mathbf{l}^*\times\mathbf{v},\ \mathbf{l}=\mathbf{l}^* \tag{3.2}$$

Below, we shall confine ourselves to the case when the instantaneous stress tensors are equal to zero $q^{\alpha\beta}=Q^{\alpha\beta}=0$. From (2.1)–(2.4) and (3.1), (3.2) we have the equation of heat flow (henceforth, terms of order v^2/c^2 will be omitted during the transformations)

$$\begin{aligned} \rho\frac{dU}{dt}=&\left(p^{\alpha\beta}-\frac{1}{2}\mathbf{E}^*\cdot\mathbf{P}^*g^{\alpha\beta}\right)e_{\alpha\beta}-\nabla_\beta q^\beta+\\ &\rho\mathbf{E}^*\cdot\left(\frac{d\boldsymbol{\pi}^*}{dt}+\boldsymbol{\pi}^*\times\boldsymbol{\omega}\right)+\rho\boldsymbol{\omega}\cdot\frac{d\mathbf{k}^*}{dt}-\rho\mathbf{a}\cdot\left(\frac{d\mathbf{l}}{dt}+\mathbf{l}\times\boldsymbol{\omega}\right) \end{aligned} \tag{3.3}$$

$$e_{\alpha\beta}=\frac{1}{2}(\nabla_\beta v_\alpha+\nabla_\alpha v_\beta),\ \boldsymbol{\omega}=\mathrm{rot}\,\mathbf{v}$$

$$U=e-\frac{v^2}{2}-\mathbf{v}\cdot(\mathbf{p}-\mathbf{v})+\frac{1}{2}\mathbf{E}^*\cdot\boldsymbol{\pi}^*,\ \boldsymbol{\pi}=\frac{1}{\rho}\mathbf{P} \tag{3.4}$$

Since the microscopic mass moment of the system is introduced in the same way as the polarization moment, we shall assume that the following relation holds:

$$\mathbf{l}^*=\boldsymbol{\pi}^*/\gamma,\ \gamma=\mathrm{const}$$

and restrict ourselves to the case of an isotropic medium, when

$$U=U_0-\frac{\partial U_0}{\partial\, d\pi^\alpha/dt}\frac{d\pi^\alpha}{dt},\ U_0=U_0\left(\rho, s, \pi^\alpha, \frac{d\pi^\alpha}{dt}, g_{\alpha\beta}\right)$$

we can find the quantities T and p from the formulas

$$T = \partial U_0/\partial S, \quad p = \rho^2 \partial U_0/\partial \rho$$

Then the equation of heat flow (3.3) yields, with the above assumptions, the equation of entropy balance

$$\rho T \frac{ds}{dt} = \left[p^{(\alpha\beta)} - \left(\frac{1}{2}\mathbf{E}^*\cdot\mathbf{P}^* - p\right) g^{\alpha\beta} \right] e_{\alpha\beta} - \nabla_\alpha q^\alpha + \tag{3.5}$$
$$\rho \mathbf{w}\cdot\left(\frac{d\boldsymbol{\pi}}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}\right) + \rho\boldsymbol{\omega}\cdot\frac{d}{dt}\left(\mathbf{k}^* - \frac{\partial U_0}{\partial\, d\boldsymbol{\pi}/dt}\times\boldsymbol{\pi}\right)$$
$$\mathbf{w} = \mathbf{E}^* - \frac{1}{\gamma}\mathbf{a} - \frac{\partial U_0}{\partial \boldsymbol{\pi}} + \frac{d}{dt}\frac{\partial U_0}{\partial\, d\boldsymbol{\pi}/dt}$$

derived with the help of the identity

$$\frac{\partial U_0}{\partial \boldsymbol{\pi}} \times \boldsymbol{\pi} + \frac{\partial U_0}{\partial\, d\boldsymbol{\pi}/dt} \times \frac{d\boldsymbol{\pi}}{dt} \equiv 0$$

which follows from the fact that the scalar function U_0 can depend on its vector arguments $\boldsymbol{\pi}$, $d\boldsymbol{\pi}/dt$ only through the convolutions $\boldsymbol{\pi}\cdot d\boldsymbol{\pi}/dt$, $\boldsymbol{\pi}^2$, $(d\boldsymbol{\pi}/dt)^2$.

If the presence of internal angular momentum in the characteristic system is connected only with the dependence of the function U_0 on the derivatives $d\pi^\alpha/dt$, i.e.

$$k^* = \frac{\partial U_0}{\partial\, d\boldsymbol{\pi}/dt} \times \boldsymbol{\pi}$$

then the last term on the right-hand side of (3.5) will vanish. Representing the entropy gain in the form /1/ $ds = d_e s + d_i s$ and specifying the external flux

$$\rho T \frac{d_e s}{dt} = -T\nabla_\beta\left(\frac{1}{T} q^\beta\right)$$

we obtain the following expression for the dissipative functional:

$$\sigma \equiv \rho T \frac{d_i s}{dt} = \left[p^{(\alpha\beta)} - \left(\frac{1}{2}\mathbf{E}^*\cdot\mathbf{P}^* - p\right) g^{\alpha\beta}\right] e_{\alpha\beta} + \tag{3.6}$$
$$T q^\beta \nabla_\beta \frac{1}{T} + \rho\mathbf{w}\cdot\left(\frac{d\boldsymbol{\pi}}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}\right)$$

Let us assume that σ is a quadratic function of thermodynamic fluxes: $e_{\alpha\beta}$, $T\nabla_\beta(1/T)$, $\mathbf{w}$ and the corresponding forces are defined by the relations

$$p^{(\alpha\beta)} - \left(\frac{1}{2}\mathbf{E}^*\cdot\mathbf{P}^* - p\right) g^{\alpha\beta} = \frac{1}{2}\frac{\partial\sigma}{\partial e_{\alpha\beta}} \tag{3.7}$$
$$\rho\left(\frac{d\boldsymbol{\pi}}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}\right) = \frac{1}{2}\frac{\partial\sigma}{\partial\mathbf{w}}, \quad q^\beta = \frac{1}{2}\frac{\partial\sigma}{\partial(T\nabla_\beta(1/T))}$$

When $\gamma = e/m$, the vector $\mathbf{E}^* - \mathbf{a}/\gamma$ represents the effective electric field acting on the polarization electrons. The action of inertia forces on the conduction electrons is known as the Stuart-Tolman effect, and can also be described by introducing the effective field vector $\mathbf{E}^* - \mathbf{a}/\gamma$ [3].

The complete system of equations of the model comprises Maxwell's equations (2.5), the equations of continuity, momentum balance (2.1), internal momentum balance (2.3), (2.4), entropy balance (3.5), and the kinetic relations (3.7). Note that using (2.4) we can transform the momentum balance equations (2.1) and moment equations (2.3) to the form

$$\rho\frac{dv^\alpha}{dt} = \nabla_\beta p^{\alpha\beta} - \frac{1}{2}\nabla^\alpha(\mathbf{P}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{M}) + \frac{\rho}{c}\frac{d}{dt}\boldsymbol{\pi}\times\mathbf{B}^\alpha + \tag{3.8}$$
$$\mathbf{P}\cdot\nabla^\alpha\mathbf{E} + \mathbf{M}\cdot\nabla^\alpha\mathbf{B} - \rho\frac{d^2 l^\alpha}{dt^2}$$
$$p_{\gamma\beta} - p_{\beta\gamma} = \rho(\pi_\gamma w_\beta - \pi_\beta w_\gamma)$$

4. Consider the problem of the propagation of weak perturbations in a charge-free, nonconducting medium possessing internal mass moment. In this case the motion of the medium will be described by a system of equations containing Maxwell's equations (2.5) with $q = 0$, $\mathbf{j} = 0$, the equations of continuity and

$$\rho\frac{dv^\alpha}{dt} = -\nabla^\alpha p + \mathbf{P}\cdot\nabla^\alpha\mathbf{E} + \mathbf{M}\cdot\nabla^\alpha\mathbf{B} + \frac{\rho}{c}\frac{d}{dt}\boldsymbol{\pi}\times\mathbf{B}^\alpha - \tag{4.1}$$
$$\frac{\rho}{\gamma}\frac{d^2\pi^\alpha}{dt^2} + \frac{1}{2}\nabla_\beta\left\{\rho\tau\left[\pi^\alpha\left(\frac{d\pi^\beta}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}^\beta\right) - \right.\right.$$
$$\left.\left.\pi^\beta\left(\frac{d\pi^\alpha}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}^\alpha\right)\right]\right\}, \quad \rho\left(\frac{d\boldsymbol{\pi}}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}\right) = \frac{\mathbf{w}}{\tau}$$
$$\rho T\frac{ds}{dt} = \rho^2\tau\left(\frac{d\boldsymbol{\pi}}{dt} + \boldsymbol{\pi}\times\boldsymbol{\omega}\right)^2, \quad \mathbf{E}^* = \mathbf{E} - \frac{1}{c}\mathbf{B}\times\mathbf{v}$$
$$\mathbf{M} = \frac{1}{c}\mathbf{P}\times\mathbf{v}, \quad p = \rho^2\frac{\partial U_0}{\partial\rho}, \quad T = \frac{\partial U_0}{\partial s}$$

In (4.1) τ is a scalar function of the state parameters, and has dimensions of time. Polarization relaxation is the only irreversible process in the medium taken into account.

Let us assume, for simplicity, that the function U_0 is given in the form

$$U_0 = U_1(\rho, s) + \frac{1}{2\varkappa}\pi\cdot\pi + \frac{\eta}{2}\frac{d\pi}{dt}\cdot\frac{d\pi}{dt} \tag{4.2}$$

($\varkappa$ and η are certain functions of the density and entropy).

We shall assume that a weak perturbation propagates through a homogeneous medium at rest, and there is no electromagnetic field in the unperturbed state.

Linearizing the equations with respect to the unperturbed state, we obtain

$$\begin{gathered}\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \quad \operatorname{div}\mathbf{D} = 0 \\ \frac{\partial\rho'}{\partial t} + \rho_0 \operatorname{div}\mathbf{v} = 0, \quad \rho_0\frac{\partial v^\alpha}{\partial t} = -a_0^2\nabla^\alpha\rho' - b_0\nabla^\alpha s' - \frac{\rho_0}{\gamma}\frac{\partial^2\pi^\alpha}{\partial t^2} \\ E^\alpha = \frac{1}{\gamma}\frac{\partial v^\alpha}{\partial t} + \frac{1}{\varkappa}\pi^\alpha - \eta\frac{\partial^2\pi^\alpha}{\partial t^2} + \rho_0\tau\frac{\partial\pi^\alpha}{\partial t}, \quad \rho_0\frac{\partial s'}{\partial t} = 0 \\ a_0^2 = \frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial U_0}{\partial\rho}\right)_{|\rho_0, s_0}, \quad b_0 = \frac{\partial}{\partial s}\left(\rho^2\frac{\partial U_0}{\partial\rho}\right)_{|\rho_0, s_0}\end{gathered} \tag{4.3}$$

Here ρ' and s' denote the perturbations in the density and entropy of the medium, $\rho' = \rho - \rho_0$; $s' = s - s_0$ (ρ_0 and s_0 denote the density and entropy in the unperturbed state).

We shall seek a solution of (4.3) in the form

$$A(x^\alpha, t) = A_1 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)] \tag{4.4}$$

(where A represents any function appearing in (4.3)). Denoting for convenience the amplitudes of the parameter perturbations by the same letters as the perturbations themselves, we obtain from (4.3) the following set of algebraic equations:

$$\begin{gathered}(\mathbf{E} + 4\pi\rho_0\pi)\cdot\mathbf{k} = 0, \quad \mathbf{H}\cdot\mathbf{k} = 0 \\ \mathbf{E} + 4\pi\rho_0\pi = -\frac{c}{\omega}\mathbf{k}\times\mathbf{H}, \quad \mathbf{H} = \frac{c}{\omega}\mathbf{k}\times\mathbf{E} \\ \omega\rho' = \rho_0\mathbf{k}\cdot\mathbf{v}, \quad \rho_0\omega\mathbf{v} = a_0^2\rho'\mathbf{k} + b_0 s'\mathbf{k} + \frac{\rho_0}{\gamma}\omega^2 i\pi \\ \mathbf{E} = -\frac{i}{\gamma}\omega\mathbf{v} + \frac{1}{\varkappa}\pi + \eta\omega^2\pi - i\rho_0\tau\omega\pi, \quad \rho_0\omega s' = 0\end{gathered} \tag{4.5}$$

Solving (4.5) for the components of the vector π we arrive (in the case when $\omega \neq 0$) at the following relation:

$$\begin{gathered}\left\{\frac{4\pi\rho_0 c^2}{\omega^2}\left[1 - \left(\frac{ck}{\omega}\right)^2\right]^{-1} - \frac{a_0^2}{\gamma^2}\left[1 - \left(\frac{a_0 k}{\omega}\right)^2\right]^{-1}\right\}(\pi\cdot\mathbf{k})\mathbf{k} = \\ \left\{4\pi\rho_0\left[1 - \left(\frac{ck}{\omega}\right)^2\right]^{-1} + \Omega\right\}\pi \\ \Omega = \frac{1}{\varkappa} + \left(\frac{1}{\gamma^2} + \eta\right)\omega^2 - i\rho_0\tau\omega\end{gathered} \tag{4.6}$$

This equation describes the fact that two types of waves may propagate through the medium.

Transverse waves: $\pi\cdot\mathbf{k} = 0$. From (4.5) it follows that in this case the perturbations amplitude vectors $\mathbf{E}, \mathbf{H}, \mathbf{v}, \pi$ are orthogonal to the wave vector $\mathbf{k}$; the density perturbation is equal to zero: $\rho' = 0$. From (4.6) we find the relations connecting the wave frequency and wave number

$$\frac{\omega^2}{k^2} = c^2\left[1 + \frac{4\pi\rho_0}{\Omega}\right]^{-1}$$

At the low frequencies ($\omega \to 0$) the above expression becomes a well-known dissipation relation for the electromagnetic waves in a dielectric /3/.

$$\omega/k = c/\sqrt{\varepsilon}, \quad \varepsilon = 1 + 4\pi\rho_0\varkappa$$

In the limit as $\omega \to \infty$, the phase velocity of propagation of transverse waves is equal to the speed of light in vacuo: $\omega/k = c$.

Longitudinal waves: $\pi \parallel \mathbf{k}$. In the longitudinal wave the perturbation amplitude vectors $\mathbf{E}, \pi, \mathbf{v}$ are parallel to the wave vector $\mathbf{k}$, and the perturbation of the magnetic field is zero $\mathbf{H} = 0$. In this case relation (4.6) yields

$$\frac{\omega^2}{k^2} = a_0^2\left[1 - \frac{\omega^2}{\gamma^2(\Omega + 4\pi\rho_0)}\right]$$

At high frequencies ($\omega \to \infty$) the above relation becomes

$$\omega^2/k^2 = a_0^2\eta\gamma^2/(\eta\gamma^2 + 1)$$

In the limit as $\omega \to 0$, we have $\omega/k = a_0$. Thus the quantity a_0 has the meaning of the "equilibrium" phase velocity of the longitudinal sound waves.

When the internal mass moment is neglected (in the limit as $\gamma \to \infty$) from (4.3) it follows that in the transverse waves the electromagnetic quantities $(\pi, \mathbf{H}, \mathbf{E})$, and in the longitudinal waves the mechanical quantities $(\mathbf{v}, \rho)$ are the only ones perturbed.

In addition to the two types of weak perturbations discussed here, we also have a solution of (4.5) corresponding to the case when $\omega = 0$. This wave does not propagate through space and represents an arbitrarily small deviation in the entropy distribution from its value in the equilibrium state. From (4.5) it follows that in the entropic wave $\pi = 0$, $\mathbf{H} = \mathbf{E} = 0$, $\mathbf{v} = 0$ and the only non-zero perturbations are those of density and entropy.

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NON-LINEAR EQUATIONS OF THE DYNAMICS OF AN ELASTIC MICROPOLAR MEDIUM*

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The non-linear equations for a continuous elastic medium with three additional degrees of freedom associated with local rotation, are considered. Such an elastic medium is called micropolar /1/. The existence of an elastic potential is proposed for it; thermal effects are neglected.

The purpose of this paper is to study certain qualtiative properties of the equations that are closely associated with the concept of hyperbolicity. The complete set of equations is represented as a system of local conservation laws, closed by finite relationships yielding the rheology of the material. The possibility of such a representation is based /2, 3/ on the fact that the gradients of the particle displacment and angle of rotation are used as a measure of the deformation. Local conservation laws for the compatibility of the strain and velocity fields of fairly simple structure are formulated.

The velocities of propgation of characteristic surfaces are studied for the dynamic equations for the general case of the material under consideration. The existence of real velocities, the necessary condition for hyperbolicity, results in a constraint on the form of the elastic potential function, which is an analog of the *SE*-inequality /4/ in the classical theory of non-linearly elastic media.

The system of non-linear equations being studied is reduced to symmetric form by replacing the vector of the solution. The necessary condition for such a transformation /5/ is the existence of an additional energy conservation law that follows from the system under consideration. The symmetric form of the equations enables us to formulate the sufficient condition for hyperbolicity - the condition of convexity of the elastic potential in its arguments. An estimate is obtained for the growth of the solutions of the Cauchy problem and the ensuing uniqueness theorem. The presence of the symmetric form of the system enables a general form to be obtained for the transport equation that governs the rate of change of a weak discontinuity along a bicharacteristic.

1. Fundamental equations.

Let $\mathbf{X}$ be the radius-vector of a material particle of a body in the reference configuration $\varkappa$. We assume that the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ and

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